Exercise 4

Use the *modified decomposition method* to solve the following Volterra integral equations:

$$u(x) = 3x^2 + (1 - e^{-x^3}) - \int_0^x e^{-x^3+t^3} u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= 3x^2 + (1 - e^{-x^3}) - \int_0^x e^{-x^3+t^3} \sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= 3x^2 + (1 - e^{-x^3}) - \int_0^x e^{-x^3+t^3} [u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{3x^2}_{u_0(x)} + \underbrace{(1 - e^{-x^3}) - \int_0^x e^{-x^3+t^3} u_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x [-e^{-x^3+t^3} u_1(t)]\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

Grouping the terms as we have makes it so that the series terminates early.

$$\begin{aligned}
u_0(x) &= 3x^2 \\
u_1(x) &= (1 - e^{-x^3}) - \int_0^x e^{-x^3+t^3} u_0(t)\,dt = (1 - e^{-x^3}) - (1 - e^{-x^3}) = 0 \\
u_2(x) &= \int_0^x [-e^{-x^3+t^3} u_1(t)]\,dt = 0 \\
&\vdots \\
u_n(x) &= \int_0^x [-e^{-x^3+t^3} u_{n-1}(t)]\,dt = 0, \quad n > 2
\end{aligned}$$

Therefore,

$$u(x) = 3x^2.$$